Homework 1

13. No

We prove it by Contradiction.

Suppose it is a vector field.

Then, there exists $\vec{0} \in V$, s.t.,

$(a_1, a_2) + \vec{0} = (a_1, a_2)$, $\forall (a_1, a_2) \in V$.

Then, we must have $\vec{0} = (0, 1)$.

However, $\forall (b_1, b_2) \in V$,

$(0, 0) + (b_1, b_2) = (b_1, 0) \neq \vec{0} = (0, 1)$,

which is a contradiction.

Hence, V is not a vector field.

21. Since V and W are vector spaces,

Z is a vector space by direct checking 8 axioms.

11. No.

Take $n = 2$,

$f_1(x) = -x + x^2$

$f_2(x) = -x - x^2$

$f_1(x) + f_2(x) = -2x$

So, $\deg(f_1+f_2) = 1 \neq 2$

19. "$\Leftarrow$" part:

Note that $W_1 \cup W_2 = \begin{cases} W_1 & \text{if } W_2 \subseteq W_1 \\ W_2 & \text{if } W_1 \subseteq W_2 \end{cases}$

Hence, $W_1 \cup W_2$ is a subspace of V.

"$\Rightarrow$" part:

We prove it by contradiction.

Suppose $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$.

Then, we can find $\vec{v}$, s.t., $v \in W_1$, $v \notin W_2$,

and $\vec{u}$, s.t., $\vec{u} \notin W_1$ $\vec{u} \in W_2$

Then, without loss of generality, suppose $(\vec{u}+\vec{v}) \in W_1$.

Then we have $\vec{u} = (\vec{u}+\vec{v}) + (-\vec{v}) \in W_1$,

which is a contradiction.

31. (a). "$\Rightarrow$" part:

Since $v+W$ is a subspace of V,

$\vec{0} \in v+W$.

Hence, $v+w = \vec{0}$ for some $w \in W$

$\Rightarrow w = -v \Rightarrow v \in W$.

"$\Leftarrow$" part:

if $v \in W$, clearly $v+W \subset W$.

Also, for any $w \in W$,

$w = v + (-v + w) \in v + W$.

Hence, $v + W = W$

$\Rightarrow v + W$ is a subspace of V.

(b) "$\Rightarrow$" part:

Since $v_1 + W = v_2 + W$,

$v_1 + \vec{0} = v_1 \in v_2 + W$

$\Rightarrow v_1 = v_2 + w$ for some $w \in W$

$\Rightarrow v_1 - v_2 = w$

$\Rightarrow v_1 - v_2 \in W$

"$\Leftarrow$" part:

for any $w \in W$,

we have $v_1 + w = v_2 + (v_1 - v_2 + w)$

$\Rightarrow v_1 + W = v_2 + W$.

(c) Note that for any $w_1, w_2 \in W$,

$$(v_1 + w_1) + (v_2 + w_2) = \left(v_1' + (v_1 - v_1' + w_1)\right) + \left(v_2' + (v_2 - v_2' + w_2)\right)$$

$\Rightarrow (v_1 + W) + (v_2 + W) = (v_1' + W) + (v_2' + W)$.

Also, for any $w \in W$, $a \in \mathbb{F}$.

$$a(v_1 + w) = a\left(v_1' + (v_1 - v_1' + w)\right)$$

$\Rightarrow a(v_1 + W) = a(v_1' + W)$.

(d) • $(v_1+W)+(v_2+W) = (v_1+v_2)+W = (v_2+W)+(v_1+W)$

• $\big((v_1+W)+(v_2+W)\big)+(v_3+W)$

$= \big((v_1+v_2)+W\big)+(v_3+W)$

$= (v_1+v_2+v_3)+W$

$= (v_1+W)+\big((v_2+v_3)+W\big)$

$= (v_1+W)+\big((v_2+W)+(v_3+W)\big)$

• for any $v\in V$,

$(v+W)+(0+W) = (v+0)+W = v+W$

• $\forall\, v\in V$,

$(v+W)+(-v+W) = 0+W$

• $1(v+W) = 1v+W = v+W$

• $(ab)(v+W) = (ab)v+W = a(bv)+W = a(bv+W) = a\big(b(v+W)\big)$

• $a\big((v+W)+(u+W)\big) = a\big((u+v)+W\big) = (au+av)+W$

$= (au+W)+(av+W) = a(u+W)+a(v+W)$

• $(a+b)(v+W) = (a+b)v+W$

$= (av+bv)+W$

$= (av+W)+(bv+W)$

$= a(v+W)+b(v+W)$

Hence, V/W is a vector space.

10. Let $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be an arbitrary symmetric 2×2 matrix,

where $a, b, c \in \mathbb{R}$.

We see, $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

14. • Suppose $u_1, \cdots, u_n \in S_1$; $v_1, \cdots, v_m \in S_2$,

$a_1, \cdots, a_n \in \mathbb{F}$; $b_1, \cdots, b_m \in \mathbb{F}$.

Then, $a_1u_1 + \cdots + a_nu_n + b_1v_1 + \cdots + b_nv_n \in \text{span}\{S_1 \cup S_2\}$

since $u_1, \cdots, u_n \in S_1 \cup S_2$, $v_1, \cdots, v_n \in S_1 \cup S_2$.

$\Rightarrow \text{span}(S_1) + \text{span}(S_2) = \text{span}(S_1 + S_2)$

• Suppose $w_1, \cdots, w_n \in S_1 \cup S_2$; $c_1, \cdots, c_n \in \mathbb{F}$.

Then, we can reorder $w_1, \cdots, w_n$ to be

$w_{i_1}, \cdots, w_{i_k}, w_{i_{k+1}}, \cdots, w_{i_n}$, s.t.,

$w_{i_1}, \cdots, w_{i_k} \in S_1$, $w_{i_{k+1}}, \cdots, w_{i_n} \in S_2$.

So, $c_1w_1 + \cdots + c_nw_n$

$= (c_{i_1}w_{i_1} + \cdots + c_{i_k}w_{i_k}) + (c_{i_{k+1}}w_{i_{k+1}} + \cdots + c_{i_n}w_{i_n})$

Hence, $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$

15. • Let $v_1, \cdots, v_n \in S_1 \cap S_2$, $a_1, \cdots, a_n \in \mathbb{F}$.

Then, $a_1v_1 + \cdots + a_nv_n \in \text{span}\{S_1\}$ since $v_1, \cdots, v_n \in S_1$,

$a_1v_1 + \cdots + a_nv_n \in \text{span}\{S_2\}$ since $v_1, \cdots, v_n \in S_2$.

$\Rightarrow \text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$

• Let $V = \mathbb{R}^2$.

$S_1 = \{(1,0), (0,1)\}$, $S_2 = \{(1,0)\}$.

Then, $\text{span}(S_1 \cap S_2) = \text{span}(\{(1,0)\}) = \text{span}(S_2)$

$\text{span}(S_1) = \mathbb{R}^2$.

$\Rightarrow \text{span}(S_1 \cap S_2) = \text{span}(S_1) \cap \text{span}(S_2)$

• Let $V = \mathbb{R}^2$.

$S_1 = \{(1,0)\}$, $S_2 = \{(2,0)\}$

Then, $S_1 \cap S_2 = \phi$, $\text{span}(S_1 \cap S_2) = \phi$.

But, $\text{span}(S_1) \cap \text{span}(S_2) = \text{span}(\{1,0\})$.

15. "$\Leftarrow$" part is clearly true by definition

"$\Rightarrow$" part:

Suppose $\{u_1, \cdots, u_n\}$ is linearly dependent.

Then, $a_1u_1 + \cdots + a_nu_n = 0$ for some $(a_1, \cdots, a_n) \neq 0$.

Let k be the largest integer in $\{1, \cdots, n\}$

satisfying $a_k \neq 0$.

If $k=1$, then we have $u_1 = 0$.

If $k>1$, then we have $a_1u_1 + \cdots + a_ku_k = 0$

$\Rightarrow a_ku_k = -(a_1u_1 + \cdots + a_{k-1}u_{k-1})$

$\Rightarrow u_k = -\frac{a_1}{a_k}u_1 - \frac{a_2}{a_k}u_2 - \cdots - \frac{a_{k-1}}{a_k}u_{k-1}$

This finishes the proof of "$\Rightarrow$" part.

18. Arbitrarily choose $\{p_1, \cdots, p_n\} \in S$.

Without loss of generality,

suppose $\deg(p_1) < \deg(p_2) < \cdots < \deg(p_n)$.

Suppose $a_1p_1 + \cdots + a_np_n = 0$ for some $a_1, \cdots, a_n \in \mathbb{F}$.

We see the coefficient of the term $x^{\deg(p_n)}$

is a_n because $\deg(p_1) < \cdots < \deg(p_{n-1}) < \deg(p_n)$.

So, $a_n = 0$ by $a_1p_1 + \cdots + a_np_n = 0$.

Then, $a_1p_1 + \cdots + a_np_n = a_1p_1 + \cdots + a_{n-1}p_{n-1}$.

Next, observe that the coefficient of the term $x^{\deg(p_{n-1})}$

is a_{n-1}.

So, $a_{n-1} = 0$ by $a_1p_1 + \cdots + a_{n-1}p_{n-1} = 0$.

Continue this process, we see $a_1 = \cdots = a_n = 0$.

Hence, S is linearly independent.